Hamiltonian pseudo-representations

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Abstract

The question studied here is the behavior of the Poisson bracket under C^0-perturbations. In this purpose, we introduce the notion of pseudo-representation and prove that the limit of a converging pseudo-representation of any normed Lie algebra is a representation.

An unexpected consequence of this result is that for many non-closed symplectic manifolds (including cotangent bundles), the group of Hamiltonian diffeomorphisms (with no assumptions on supports) has no C^{-1} bi-invariant metric. Our methods also provide a new proof of Gromov-Eliashberg Theorem, it is to say that the group of symplectic diffeomorphisms is C^0-closed in the group of all diffeomorphisms.

1 Statement of results

1.1 Poisson Brackets and C^0-convergence

We consider a symplectic manifold (M, ω). A function H on M will be said normalized if $\int_M H\omega^n = 0$ for M closed or if H has compact support otherwise. We will denote $C_0^\infty(M)$ the set of normalized smooth functions. Endowed with the Poisson brackets $\{\cdot, \cdot\}$, it has the structure of a Lie algebra.

In the whole paper, we will denote X_H the symplectic gradient of a smooth function H, i.e., the only vector field satisfying $dH = \iota_{X_H}\omega$. Then, the Poisson brackets are given by $\{H, K\} = dH(X_K)$.

Let $\mathfrak{g}$ be a normed Lie algebra, i.e., a Lie algebra endowed with a norm $\|\cdot\|$ such that for some constant C,

$$\|[f,g]\| \leqslant C\|f\|\cdot\|g\|,$$

and consider the following definition.

Definition 1. *A sequence of linear maps*

$$\rho_n : (\mathfrak{g}, \|\cdot\|) \to (C_0^\infty(M), \|\cdot\|_{C^0}),$$

will be called a pseudo-representation if the sequence of bilinear maps

$$B_n : (f,g) \mapsto \{\rho_n(f), \rho_n(g)\} - \rho_n([f,g])$$

converges to 0.

If it has a limit, we may ask whether this limit is a representation. If so, we would have

$$\{\rho_n(f), \rho_n(g)\} \to \{\rho(f), \rho(g)\}, \text{ for all } f, g \in \mathfrak{g}.$$

This has been proved in [1] for abelian Lie algebras. The main result of this paper is that it holds for all normed Lie algebras.

Theorem 2. *For any normed Lie algebra (in particular for finite dimensional Lie algebras), the limit of a converging pseudo-representation is a representation.*

Remark 1. This result generalizes Gromov-Eliashberg's Theorem of C^0 closure of the symplectomorphisms group in the group of diffeomorphisms.

Indeed, a diffeomorphism of $\mathbb{R}^{2n}$ is symplectic if and only if its coordinate functions $(f_i), (g_i)$ satisfy

$$\{f_i, g_j\} = \delta_{ij}, \quad \{f_i, f_j\} = \{g_i, g_j\} = 0.$$

Thus we can easily see that a sequence of symplectomorphisms gives a pseudo-representation of a 2-nilpotent Lie algebra. If the support of the coordinate functions were compact, we could immediately apply Theorem 2. In fact, for compactly supported symplectomorphisms, these functions are affine at infinity, and we have to adapt the proof to this case (See Appendix A for details).

Remark 2. Consider the following question: If F_n, G_n and $\{F_n, G_n\}$ respectively converge to F, G and H (all function being smooth and normalized, and all convergence being in the C^0 sense), is it true that $\{F, G\} = H$?

Theorem 2 states that the answer is positive when there is some Lie algebra structure. Nevertheless, in general, the answer is negative, as shows the following example, which is derived from Polterovich's example presented in Section 2.3. Let χ be a compactly supported smooth function on $\mathbb{R}$, and set the following functions on $\mathbb{R}^2$:

$$F_n(q,p) = \frac{\chi(p)}{\sqrt{n}} \cos(nq),$$

$$G_n(q,p) = \frac{\chi(p)}{\sqrt{n}} \sin(nq).$$

It is easy to see that F_n and G_n converge to 0, but that their Poisson brackets equal $\chi(p)\chi'(p) \neq 0$.

This example shows that when the Poisson brackets C^0-converge, then its limit is not necessarily the brackets of the respective limits. But in that case, we can see that the Hamiltonians F_n and G_n do not generate a pseudo-representation.

Remark 3. The theorem holds if we replace the symplectic manifold with a general Poisson manifold. Indeed, Poisson manifolds are foliated by Poisson submanifolds that are symplectic, and we just have to apply theorem 2 to each leaf.

Remark 4. The theorem leads us to the following

Definition 3. *A* continuous Hamiltonian representation *of a normed Lie algebra* $\mathfrak{g}$ *is a continuous linear map* $\mathfrak{g} \to C^0(M)$ *which is the* C^0*-limit of some pseudo-representation of* $\mathfrak{g}$*.*

We will not study this notion further in this paper. Nevertheless let us give some example:

Example: Let $\rho : \mathfrak{g} \to C_0^\infty(M)$ be a smooth Hamiltonian representation in the usual sense, and let φ be a homeomorphism of M which is the C^0-limit of a sequence of symplectomorphisms. Then, $\rho' : \mathfrak{g} \to C^0(M)$, given by $\rho'(g) = \rho(g) \circ \varphi$, is clearly a continuous Hamiltonian representation.

Question 1: Given two sequences of Hamiltonians (F_n), (G_n) that C^0-converge to smooth F and G, is there some sufficient condition for the bracket $\{F,G\}$ not to be the limit of the brackets $\{F_n, G_n\}$? Propositions 12 and 13 give restrictions on the possible counter- examples.

Question 2: Let us consider the following number introduced by Entov, Polterovich and Zapolsky in [2]:

$$\Upsilon(F,G) = \lim_{\varepsilon \to 0} \inf \{ \|\{F', G'\}\| \mid \|F - F'\|_{C^0} < \varepsilon, \|G - G'\|_{C^0} < \varepsilon \}$$

The result of Cardin and Viterbo mentioned above which is exactly Theorem 2 in the abelian case can be restated as follows:

$$\Upsilon(F,G) > 0 \text{ if and only if } \{F,G\} \neq 0.$$

Entov, Polterovich and Zapolsky have improved this result by giving explicit lower bounds on $\Upsilon(F,G)$, in terms of quasi-states (see [2] and [16]). We may wonder whether there exist similar inequalities in the non abelian case.

1.2 Bi-invariant Metrics

Here we consider a subgroup $\mathcal{G}$ of the group $\mathcal{H}(M)$ of Hamiltonian diffeomorphisms on M. If we denote ϕ_H^t the flow generated by X_H (when it exists), and $\phi_H = \phi_H^1$ the time-1 map, $\mathcal{H}(M)$ is the set of all diffeomorphisms ϕ for which it exists a path of Hamiltonian functions $H_t \in C^\infty(M)$ such that $\phi = \phi_H$.

Definition 4. *A* bi-invariant metric *on $\mathcal{G}$ is a distance d on $\mathcal{G}$ such that for any ϕ, ψ, χ in $\mathcal{G}$,*

$$d(\phi,\psi) = d(\phi\chi,\psi\chi) = d(\chi\phi,\chi\psi).$$

It will be said C^{-1} if its composition with the map $\Phi : H \mapsto \phi_H^1$ is a continuous map $\Phi^{-1}(\mathcal{G}) \times \Phi^{-1}(\mathcal{G}) \to \mathbb{R}$, where $\Phi^{-1}(\mathcal{G}) \subset Ham$ is endowed with the compact-open topology.

There are several well known examples of C^{-1} bi-invariant metrics, as, for example, Hofer's metric defined on the subgroup Hamiltonian diffeomorphisms generated by compactly supported functions $\mathcal{H}_c(M)$ (see [4] or [7]), Viterbo's metric defined on $\mathcal{H}_c(\mathbb{R}^{2n})$ (see [15]), and its analogous version defined by Schwarz in [12] for symplectically aspherical closed symplectic manifolds.

As far as we know, if we remove the assumption of compactness of the support, the question whether there exists such metrics is still open. Here we prove that the answer is negative for a large class of symplectic manifolds.

Let (N,ξ) be a contact manifold with contact form α (i.e., a smooth manifold N with a smooth hyperplane section ξ which is locally the kernel of a 1-form α whose differential $d\alpha$ is non-degenerate on ξ). Its *symplectization* is by definition the symplectic manifold $\mathcal{S}N = \mathbb{R} \times N$ endowed with the symplectic form $\omega = d(e^s\alpha)$, where s denotes the $\mathbb{R}$-coordinate in $\mathbb{R} \times N$. For any contact form α, one can define the *Reeb vector field* X_R by the identities $\iota_{X_R} d\alpha$ and $\alpha(X_R) = 1$. The trajectories of X_R are called *characteristics*. The question of the existence of a closed characteristic constitutes the famous

Weinstein's conjecture. It has now been proved for large classes of contact manifolds (see e.g. [3, 5, 6, 11, 10, 14, 13]...).

Let us now state our result that will be proved in section 2.3

Theorem 5. *If M is the symplectization of a contact manifold whose dimension is at least 3 and that admits a closed characteristic, then there is no C^{-1} bi-invariant metric on $\mathcal{H}(M)$.*

Corollary 6. *If N is a smooth manifold whose dimension is at least 2 and if T^*N is its cotangent bundle, then there is no C^{-1} bi-invariant metric on $\mathcal{H}(T^*N)$.*

Remark. At least in the case of manifolds of finite volume, there probably exists non closed manifolds with such distances. Indeed, it follows from our previous work [8] that Viterbo's metric extends to Hamiltonians functions smooth out of a "small" compact set. Replacing Viterbo's metric with Schwarz's metric, we can reasonably expect to have: If M^{2n} is a closed symplectically aspherical manifold and K is a closed submanifold of dimension $\leqslant n-2$, then Schwarz's metric on $\mathcal{H}(M)$ extends to $\mathcal{H}(M-K)$.

2 Proofs

2.1 Identities for Hamiltonian pseudo-representations

Lemma 7. *Let ρ_n be a bounded (not necessarily converging) pseudo-representation of a normed Lie algebra $\mathfrak{g}$. Let $f, g \in \mathfrak{g}$, then the sequence of Hamiltonian functions*

$$\rho_n(f) \circ \phi^s_{\rho_n(g)} - \sum_{j=0}^{+\infty} \rho_n(ad(g)^j f)\, \frac{s^j}{j!}$$

converges to zero for the C^0-norm on M. Moreover, the convergence is uniform over the s's in any compact interval.

Remark: For a representation equality holds. It recalls the Baker-Campbell-Haussdorf formula.

Proof: First remark that the considered sum converges. Indeed, the C^0-norm of its remainder can be bounded by the remainder of a converging sum, as follows:

$$\left\| \sum_{j=N}^{+\infty} \rho_n(ad(g)^j f)\, \frac{s^j}{j!} \right\| \leqslant \sum_{j=N}^{+\infty} R\|f\| \frac{(sC\|g\|)^j}{j!}.$$

where R is an n-independent upper bound for the sequence

$$\|\rho_n\| = \sup\{\|\rho_n(h)\|_{C^0} \mid \|h\| = 1\}.$$

Now, let us prove our lemma. Poisson equation gives

$$\frac{d}{ds}(\rho_n(f)\circ\phi^s_{\rho_n(g)}) = \{\rho_n(f),\rho_n(g)\}\circ\phi^s_{\rho_n(g)}$$

and hence

$$\rho_n(f)\circ\phi^{s_0}_{\rho_n(g)} = \rho_n(f) + \int_0^{s_0}\{\rho_n(f),\rho_n(g)\}\circ\phi^{s_1}_{\rho_n(g)}\,ds_1$$

$$= \rho_n(f) + \int_0^{s_0}\rho_n([f,g])\circ\phi^{s_1}_{\rho_n(g)}\,ds_1 + \int_0^{s_0}B_n(f,g)\circ\phi^{s_1}_{\rho_n(g)}\,ds_1.$$

Then, by a simple induction, we get for all integer N:

$$\rho_n(f)\circ\phi^{s_0}_{\rho_n(g)} = \sum_{j=0}^{N}\rho_n(ad(g)^j f)\,\frac{{s_0}^j}{j!} \;+\; R_{N,n}(s_0) \;+\; S_{N,n}(s_0),$$

where,

$$R_{N,n}(s_0) = \int_0^{s_0}\int_0^{s_1}\cdots\int_0^{s_N}\rho_n(ad(g)^{N+1}f)\circ\phi^{s_{N+1}}_{\rho_n(g)}ds_{N+1}\cdots ds_1$$

$$S_{N,n}(s_0) = \sum_{j=0}^{N}\int_0^{s_0}\int_0^{s_1}\cdots\int_0^{s_j}B_n(ad(g)^j f,g))\circ\phi^{s_{j+1}}_{\rho_n(g)}ds_{j+1}\cdots ds_1$$

Let us now denote

$$\|B_n\| = \sup\{\|\{\rho_n(f),\rho_n(g)\} - \rho_n([f,g])\|_{C^0} \mid \|f\| = \|g\| = 1\}.$$

By assumptions $\|B_n\|$ converges to 0.

Then,

$$\begin{aligned}\|R_{N,n}(s_0)\|_{C^0} &\leqslant \int_0^{s_0}\int_0^{s_1}\cdots\int_0^{s_{N-1}}R\|g\|^N C^N\|f\|ds_N\cdots ds_1,\\ &\leqslant R\|f\|\frac{\|g\|^N C^N s_0^N}{N!},\end{aligned}$$

which proves that $R_{N,n}(s_0)$ converges to 0 with N, uniformly in n.

In addition,

$$\|S_{N,n}(s_0)\| \leqslant \sum_{j=0}^{N-2} \int_0^{s_0} \int_0^{s_1} \cdots \int_0^{s_j} \|B_n\| \|f\| \|g\|^j ds_{j+1} \cdots ds_1$$

We thus have $\|S_{N,n}(s_0)\| \leqslant \|B_n\| \, \|f\| \exp(s_o\|g\|)$ for any N. As a consequence, letting N converge to $+\infty$, we get

$$\left\| \rho_n(f) \circ \phi^s_{\rho_n(g)} - \sum_{j=0}^{+\infty} \rho_n(ad(g)^j f) \, \frac{s^j}{j!} \right\| \leqslant \|B_n\| \, \|f\| \exp(s_o\|g\|).$$

This achieves the proof because the right hand side converges to 0. $\square$

2.2 Proof of theorem 2

Let $f, g \in \mathfrak{g}$. We want to prove that $\{\rho(f), \rho(g)\} = \rho([f,g])$. We can assume without loss of generality that $\|g\| < 1$.

By Lemma 7,

$$\rho_n(f) \circ \phi^s_{\rho_n(g)} - \sum_{j=0}^{+\infty} \rho_n(ad(g)^j f) \, \frac{s^j}{j!} \overset{C^0}{\to} 0.$$

Each term of the sum converges with n. Since the sum converges uniformly in n, we get that for any s,

$$\rho_n(f) \circ \phi^s_{\rho_n(g)} \overset{C^0}{\to} \sum_{j=0}^{+\infty} \rho(ad(g)^j f) \, \frac{s^j}{j!}.$$

As a consequence, the flow generated by $\rho_n(f) \circ \phi^s_{\rho_n(g)}$ γ-converges to the flow generated by $\sum_{j=0}^{+\infty} \rho(ad(g)^j f) \, \frac{s^j}{j!}$.

But on the other hand, the flow of $\rho_n(f) \circ \phi^s_{\rho_n(g)}$ is $t \mapsto \phi^{-s}_{\rho_n(g)} \phi^t_{\rho_n(f)} \phi^s_{\rho_n(g)}$, which γ-converges to $\phi^{-s}_{\rho(g)} \phi^t_{\rho(f)} \phi^s_{\rho(g)}$. Indeed, $\rho_n(g) \overset{C^0}{\to} \rho(g)$ and $\rho_n(f) \overset{C^0}{\to} \rho(f)$ which implies that there respective flow γ-converges.

Therefore, $t \mapsto \phi^{-s}_{\rho(g)} \phi^t_{\rho(f)} \phi^s_{\rho(g)}$ is the flow of $\sum_{j=0}^{+\infty} \rho(ad(g)^j f) \, \frac{s^j}{j!}$. The functions being normalized,

$$\rho(f) \circ \phi^s_{\rho(g)} = \sum_{j=0}^{+\infty} \rho(ad(g)^j f) \, \frac{s^j}{j!}.$$

Now, first taking derivative with respect to s, we get $\{\rho(f), \rho(g)\} = \rho([f,g])$. $\square$

2.3 Proof of theorem 5

Let us consider the following Hamiltonian functions on $\mathbb{R}^2$ (this example is due to Polterovich) with symplectic form written in polar coordinates $rdr \wedge d\theta$.

$$F_n(r,\theta) = \frac{r}{\sqrt{n}} \cos(n\theta),$$

$$G_n(r,\theta) = \frac{r}{\sqrt{n}} \sin(n\theta).$$

We see that $\{F_n, G_n\} = 1$ and that F_n and G_n converge to 0. Now, consider $\mathfrak{g}$ the 3-dimensional Heisenberg Lie algebra (i.e., the Lie algebra with basis $\{f, g, h\}$ such that $[f, g] = h$ and $[f, h] = [g, h] = 0$) and set $\rho_n(f) = F_n$, $\rho_n(g) = G_n$ and $\rho_n(h) = 1$. Then, ρ_n is a pseudo-representation of $\mathfrak{g}$ in $Ham(R^2)$. The limit ρ of ρ_n satisfies $\rho(f) = 0$, $\rho(g) = 0$, $\rho(h) = 1$. Since $\{\rho(f), \rho(g)\} \neq \rho(h)$, ρ is not a representation of $\mathfrak{g}$.

Since $\mathfrak{g}$ has finite dimension, this example shows that Theorem 2 is false in general if we replace $C_0^\infty(M)$ with $C^{infty}(M)$ for a non-compact manifold M, and uniform convergence with the uniform convergence on compact sets (compact-open topology).

If we read carefully the proof of Theorem 2, we see that the whole proof can be repeated in this settings except the three following points where the compactness of supports are needed

- Each time we consider the flows of the Hamiltonians, they must be complete. This is automatic for compactly supported Hamiltonians, but false in general. With the notations of the proof, the flows needed are those of $\rho_n(f)$, $\rho(f)$, $\rho_n(g)$, $\rho(g)$ and $\sum_{j=0}^{+\infty} \rho(ad(g)^j f) \frac{s^j}{j!}$.
- The functions $\rho_n(f)$, $\rho(f)$, $\rho_n(g)$, $\rho(g)$ have to be normalized in some sense.
- We use a C^{-1} bi-invariant metric. This exists on $\mathcal{H}_c(M)$, but we do not know whether it exists on $\mathcal{H}(M)$.

The following lemma follows from the above discussion.

Lemma 8. *Let M be a non-compact symplectic manifold, $\mathfrak{g}$ a normed Lie algebra, and ρ_n a pseudo-representation of $\mathfrak{g}$ in $Ham(M)$, with limit ρ. Suppose there exists two elements f and g in $\mathfrak{g}$, such that:*

- *all the Hamiltonian functions $\rho_n(f)$, $\rho(f)$, $\rho_n(g)$, $\rho(g)$ and $\sum_{j=0}^{+\infty} \rho(ad(g)^j f) \frac{s^j}{j!}$ exist and have complete flows,*

- *there exists an open set on which all the functions $\rho_n(f)$, $\rho(f)$, $\rho_n(g)$, $\rho(g)$ vanish identically.*

- $\{\rho(f), \rho(g)\} \neq \rho([f, g])$.

Then the group of Hamiltonian diffeomorphisms $\mathcal{H}(M)$ admits no C^{-1} bi-invariant metric. $\square$

Proof of Theorem 5: We want to apply Lemma 8. We first consider the case of $\mathbb{S}^1$. In that case we are not able to get the second requirement of Lemma 8, but let us show how we get the others.

We just adapt Polterovich's example by setting :

$$\rho_n(f)(s, \theta) = \frac{e^{s/2}}{\sqrt{n}} \cos(n\theta),$$

$$\rho_n(g)(s, \theta) = \frac{e^{s/2}}{\sqrt{n}} \sin(n\theta).$$

The symplectic form being defined on $\mathbb{R} \times \mathbb{S}^1$ by $d(e^s d\theta) = e^s ds \wedge d\theta$, we get $\{\rho_n(f), \rho_n(g)\} = 2$. Since $\rho(f) = \rho(g) = 0$ we have a pseudo-representation of the 3-dimensional Heisenberg Lie algebra, and its limit is not a representation. We can also verify that all elements $\rho_n(f)$, $\rho(f)$, $\rho_n(g)$, $\rho(g)$ and $\sum_{j=0}^{+\infty} \rho(ad(g)^j f) \frac{s^j}{j!}$ exist and have complete flows for f, g generators of the 3-dimensional Heisenberg Lie algebra, and ρ_n, ρ as in the example.

Since $\rho(f) = 0$, $\rho(g) = 0$ and $\sum_{j=0}^{+\infty} \rho(ad(g)^j f) \frac{s^j}{j!} = 2s$, this is obvious for them.

The Hamiltonian vector field of $\rho_n(f)$ is

$$\left(e^{-s/2}\sqrt{n} \sin(n\theta)\right) \frac{\partial}{\partial\theta} - \left(\frac{1}{2\sqrt{n}} e^{-s/2} \cos(n\theta)\right) \frac{\partial}{\partial s},$$

which is equivalent through the symplectomorphism

$$(\mathbb{R} \times \mathbb{S}^1, d(e^s d\theta)) \to (\mathbb{R}^2 - \{0\}, r dr \wedge d\theta)), \ (s, \theta) \mapsto (e^{-s/2}, \theta)),$$

to the vector field

$$\left(r\sqrt{n} \sin(n\theta)\right) \frac{\partial}{\partial\theta} + \left(\frac{1}{\sqrt{n}} \cos(n\theta)\right) \frac{\partial}{\partial r}.$$

The norm of this vector field is bounded by a linear function in r. Therefore, it is a consequence of Gronwall's lemma that it is complete.

Let us consider now the case $d = \dim(N) \geqslant 3$. There, we will be able to get all the requirements of Lemma 8. Denote by γ a closed characteristic, parameterized by $\theta \in \mathbb{S}^1$. Since the Reeb vector field is transverse to the contact structure ξ, there exists a diffeomorphism that maps a neighborhood $\mathcal{V}_0$ of the zero section in the restricted bundle $\xi|_\gamma$, onto a neighborhood $\mathcal{V}_1$ of γ in the contact manifold N. Since $\xi|_\gamma$ is a symplectic bundle over $\mathbb{S}^1$, it is trivial. We thus have a neighborhood U of 0 in $\mathbb{R}^{2n}$ and a diffeomorphism $\psi : \mathbb{S}^1 \times U \to \mathcal{V}_1 \subset N$. The pull back of ξ by ψ is a contact structure on $\mathbb{S}^1 \times U$ which is contactomorphic (via Moser's argument) to the standard contact structure $d\theta - pdq$ on $\mathbb{S}^1 \times U$. Therefore, the above diffeomorphism ψ can be chosen as a contactomorphism.

Then the symplectization $\mathcal{S}\gamma$ of the closed characteristic gives a symplectic embedding $\mathcal{S}\mathbb{S}^1 \hookrightarrow \mathcal{S}N$. This embedding admits $\mathcal{S}(\mathbb{S}^1 \times U)$ as a neighborhood. Moreover, if we denote s, θ and x the coordinates in $\mathcal{S}(\mathbb{S}^1 \times U)$, ψ has been constructed so that s and θ are conjugated variables and the direction of x is symplectically orthogonal to those of s and θ. That will allow the following computations.

Just like in the above example, we have a pseudo-representation of $\mathfrak{g}$ if we consider

$$(\rho_n(f))(s,\theta,x) = \frac{\chi(x)e^{s/2}}{\sqrt{n}}\cos(n\theta),$$

$$(\rho_n(g))(s,\theta,x) = \frac{\chi(x)e^{s/2}}{\sqrt{n}}\sin(n\theta), \tag{1}$$

and $(\rho_n(h))(s,\theta,x) = 2\chi(x)^2$. Indeed, we have again $\{\rho_n(f), \rho_n(g)\} = \rho_n(h)$, but its limit ρ satisfies $\{\rho(f), \rho(g)\} = 0 \neq 1 = \rho(h)$ and is not a representation. The fact that the elements $\rho_n(f)$, $\rho(f)$, $\rho_n(g)$, $\rho(g)$ and $\sum_{j=0}^{+\infty} \rho(ad(g)^j f)\,\frac{s^j}{j!}$ exist and have complete flows follows from the case $d = 1$. $\square$

Proof of Corollary 6 Let M be a smooth manifold, and choose a Riemannian metric on it. Then, consider the symplectization $\mathcal{S}\mathbb{S}T^*M$ of the sphere cotangent bundle $\mathbb{S}T^*M$. The cotangent bundle can be seen as the compactification of $\mathcal{S}\mathbb{S}T^*M$, the set at infinity being the zero section of T^*M (or $\{-\infty\} \times \mathbb{S}T^*M$ if we see $\mathcal{S}\mathbb{S}T^*M$ as $\mathbb{R} \times \mathbb{S}T^*M$).

The Reeb flow of $\mathbb{S}T^*M$ projects itself to the geodesic flow on M, and the closed characteristics are exactly the trajectories that project themselves to closed geodesics. Since any closed manifold carries a closed geodesic(see [9]), we can consider Example (1). It clearly extends to the compactification (the Hamiltonian functions involved and all their derivatives converges to 0 when s goes to $-\infty$), and we can achieve the proof as for Theorem 5. $\square$

A A proof of Gromov-Eliashberg theorem.

In this section, we show how our methods allow to recover Gromov-Eliashberg Theorem.

Theorem 9 (Gromov, Eliashberg). *The group of compactly supported symplectomorphisms $Symp_c(\mathbb{R}^{2n})$ is C^0-closed in the group of all diffeomorphisms of $\mathbb{R}^{2n}$.*

Proof. Let ϕ_n be a sequence of diffeomorphisms that converges uniformly to a diffeomorphism ϕ. Denote $(f_i^n), (g_i^n)$ (resp. f_i, g_i) the coordinate functions of ϕ_n (resp. ϕ). These coordinate functions can be seen has Hamiltonian functions affine at infinity (i.e., that can be written $H + u$ with $H \in Ham_c$ and u affine map). Moreover, for a given sequence (f_i^n) or (g_i^n), the linear part does not depend on n.

Since ϕ_n is symplectic, we have:

$$\{f_i^n, g_j^n\} = \delta_{ij}, \quad \{f_i^n, f_j^n\} = \{g_i^n, g_j^n\} = 0.$$

Thus the coordinate functions of ϕ_n give a pseudo-representation of the 2-nilpotent Lie algebra $\mathfrak{g}$ generated by elements a_i, b_i, c, with the relations

$$[a_i, b_j] = \delta_{ij},\ [a_i, a_j] = [b_i, b_j] = 0, \text{and } [a_i, c] = [b_i, c] = 0.$$

Since ϕ is symplectic if and only if

$$\{f_i, g_j\} = \delta_{ij}, \quad \{f_i, f_j\} = \{g_i, g_j\} = 0$$

the proof will be achieved if we prove that the limit of this pseudo-representation is a representation. Consequently, we have to adapt the proof of Theorem 2 to the case of Hamiltonian functions affine at infinity, for 2-nilpotent Lie algebras. Gromov-Eliashberg Theorem then follows from the next two lemmas.

Lemma 10. *Let u, v be two affine maps $\mathbb{R}^{2n} \to \mathbb{R}$ and H_n, K_n be compactly supported Hamiltonians, such that*

$$H_n \to H,\ K_n \to K,\ \{H_n + u, K_n + v\} \to 0.$$

Then $\{H + u, K + v\} = 0$.

Lemma 11. *Let u, v, w be linear forms on $\mathbb{R}^{2n}$, and H_n, K_n, G_n, be compactly supported Hamiltonians such that*

$$H_n \to H,\ K_n \to K,\ G_n \to G,$$

$$\{H_n + u, G_n + w\} \to 0,$$
$$\{K_n + v, G_n + w\} \to 0,$$
$$\{H_n + u, K_n + v\} - (G_n + w) \to 0.$$

Then $\{H+u, G+w\} = 0$, $\{K+v, G+w\} = 0$ *and* $\{H+u, K+v\} = G+w$.

Let us consider a C^{-1} biinvariant distance γ on $\mathcal{H}_c(\mathbb{R}^{2n})$ which is invariant under the action of affine at infinity Hamiltonians (such a condition is clearly satisfied by Hofer's distance). For a sequence of Hamiltonian functions that are affine at infinity with the same affine part, we can speak of its limit for γ by setting:

$$(\phi_{H_n+u}) \xrightarrow{\gamma} \phi_{H+u} \text{ if and only if } \gamma((\phi_{H+u})^{-1}\phi_{H_n+u}, Id) \to 0.$$

Moreover, if $(\phi_{H_n+u}) \xrightarrow{\gamma} \phi_{H+u}$ and $(\phi_{K_n+v}) \xrightarrow{\gamma} \phi_{K+v}$ then

$$(\phi_{H_n+u}\phi_{K_n+v}) \xrightarrow{\gamma} \phi_{H+u}\phi_{K+v}.$$

Indeed, we have

$$\begin{aligned}
&\gamma((\phi_{H_n+u}\phi_{K_n+v})^{-1}(\phi_{H+u}\phi_{K+v}), Id) \\
&\quad= \gamma(\phi_{K+v}^{-1}(\phi_{H+u}^{-1}\phi_{H_n+u})\phi_{K+v}(\phi_{K+v}^{-1}\phi_{K_n+v}), Id) \\
&\quad\leqslant \gamma(\phi_{H+u}^{-1}\phi_{H_n+u}, Id) + \gamma(\phi_{K+v}^{-1}\phi_{K_n+v}, Id).
\end{aligned}$$

Finally notice that if $\|H_n - H\|_{C^0} \to 0$, then $\phi_{H_n+u} \xrightarrow{\gamma} \phi_{H+u}$.

We are now ready for our proofs.

Proof of lemma 10. We just adapt the proof of Cardin and Viterbo [1] to the "affine at infinity" case.

First remark that the assumptions imply $\{u, v\} = 0$. Then, a simple computation shows that the flow

$$\psi_n^t = \phi_{H_n+u}^t \phi_{K_n+v}^s \phi_{H_n+u}^{-t} \phi_{K_n+v}^{-s}$$

is generated by the Hamiltonian function affine at infinity

$$\int_0^s \{H_n + u, K_n + v\}(\phi_{K_n+v}^{\sigma}\phi_{H_n+u}^t(x))d\sigma,$$

which C^0-converges to $0 = \{u, v\}$ by assumption. Therefore, ψ_n^t converges for any s and any t to Id. But on the another hand, according to the above remark, it converges to $\phi_{H+u}^t\phi_{K+v}^s\phi_{H+u}^{-t}\phi_{K+v}^{-s}$. Hence $\phi_{H+u}^t\phi_{K+v}^s\phi_{H+u}^{-t}\phi_{K+v}^{-s} = Id$ which proves $\{H+u, K+v\} = 0$. $\square$

Proof of lemma 11. First notice that the assumptions imply $\{u,v\} = w$, $\{u,w\} = 0$ and $\{v,w\} = 0$, and that the equalities $\{H+u, G+w\} = 0$, $\{K+v, G+w\} = 0$ follow from lemma 10. Here we consider the flow

$$\psi_n^t = \phi_{G_n+w}^{-ts}\phi_{H_n+u}^{t}\phi_{K_n+v}^{s}\phi_{H_n+u}^{-t}\phi_{K_n+v}^{-s}$$

which is generated by

$$\left(-s(G_n+w) + \int_0^s \{H_n+u, K_n+v\}(\phi_{K_n+v}^{\sigma}\phi_{H_n+u}^{t})d\sigma\right)\circ \phi_{G_n+w}^{ts}.$$

This expression can be written

$$\left(\int_0^s (A_n+B_n)d\sigma\right)\circ \phi_{G_n+w}^{ts},$$

where $A_n = G_n - G_n(\phi_{K_n+v}^{\sigma}\phi_{H_n+u}^{t})$ and $B_n = (\{H_n+u, K_n+v\} - (G_n+w))(\phi_{K_n+v}^{\sigma}\phi_{H_n+u}^{t})$.

By assumption, B_n C^0-converges to 0 and A_n can be written:

$$\begin{aligned}
A_n &= (G_n - G_n(\phi_{H_n+u}^{t})) + (G_n - G_n(\phi_{K_n+v}^{\sigma}))\circ\phi_{H_n+u}^{t}\\
&= \int_0^t \{G_n, H_n+u\}d\tau + \left(\int_0^\sigma \{G_n, K_n+v\}d\tau\right)\circ \phi_{H_n+u}^{t}\\
&= \int_0^t \{G_n+w, H_n+u\}d\tau + \left(\int_0^\sigma \{G_n+w, K_n+v\}d\tau\right)\circ \phi_{H_n+u}^{t},
\end{aligned}$$

which implies that A_n C^0-converges to 0 too. It follows that the generating Hamiltonian of ψ_n^t C^0-converges to 0, and hence that ψ_n^t γ-converges to Id. Since it also converges to $\psi^t := \phi_{G+w}^{-ts}\phi_{H+u}^{t}\phi_{K+v}^{s}\phi_{H+u}^{-t}\phi_{K+v}^{-s}$, we get $\psi^t = Id$ for any s and t. Thus, the generating Hamiltonian of ψ_t vanishes identically:

$$\left(-s(G+w) + \int_0^s \{H+u, K+v\}(\phi_{K+v}^{\sigma}\phi_{H+u}^{t})d\sigma\right)\circ \phi_{G+w}^{ts} = 0.$$

But since $G+w$ commutes with $H+U$ and $K+v$, we get:

$$\int_0^s (\{H+u, K+v\} - (G+w))(\phi_{K+v}^{\sigma}\phi_{H+u}^{t})d\sigma = 0.$$

Taking derivative with respect to s, we obtain $\{H+u, K+v\} - (G+w) = 0$. $\square$

B Few additional remarks using the theory of distributions.

The following results on Poisson brackets are obtained with the help of distributions. No assumptions are made on the Lie algebra generated by the Hamiltonian functions. They show in a certain way why it is difficult to find examples of pseudo-representations whose limit is not a representation.

Proposition 12. *If F_n C^2-converges to F and G_n C^0-converges to G. Then, $\{F_n, G_n\}$ converges to $\{F, G\}$ in the sense of distributions. As a consequence, if $\{F_n, G_n\}$ C^0-converges to H, then $\{F, G\} = H$.*

Proof. For any smooth compactly supported function ϕ,

$$\begin{aligned}\langle\{F_n, G_n\}, \phi\rangle &= \int \frac{\partial G_n}{\partial q}\frac{\partial F_n}{\partial p}\phi - \int \frac{\partial G_n}{\partial p}\frac{\partial F_n}{\partial q}\phi \\ &= -\int G_n \frac{\partial}{\partial q}\left(\frac{\partial F_n}{\partial p}\phi\right) + \int G_n \frac{\partial}{\partial p}\left(\frac{\partial F_n}{\partial q}\phi\right).\end{aligned}$$

By assumption, the integrands C^0-converge and hence the integrals converge to $-\int G\frac{\partial}{\partial q}\left(\frac{\partial F}{\partial p}\phi\right) + \int G\frac{\partial}{\partial p}\left(\frac{\partial F}{\partial q}\phi\right)$ which equals $\langle\{F, G\}, \phi\rangle$. $\square$

Proposition 13. *If F_n C^0-converges to F, G_n C^0-converges to G and $\{F_p, G_q\}$ C^0-converges to H when p and q go to infinity, then $\{F, G\} = H$.*

Proof. Take once again a compactly supported smooth function ϕ. Write

$$\langle\{F_p, G_q\} - \{F, G\}, \phi\rangle = \langle\{F_p - F, G_q\}, \phi\rangle + \langle\{F, G_q - G\}, \phi\rangle.$$

By Proposition 12, the first term converges to 0. Hence for all $\varepsilon > 0$, there exists an integer q_0 such that for any $q > q_0$, $|\langle\{F, G_q - G\}, \phi\rangle| \leqslant \varepsilon$.

Similarly, for each fixed q, there exists an integer p_0 such that for any $p > p_0$, $|\langle\{F_p - F, G_q\}, \phi\rangle| \leqslant \varepsilon$.

Therefore, for all ε and all integers p_1, q_1, we can find $p > p_1, q > q_1$ such that $|\langle\{F_p, G_q\} - \{F, G\}, \phi\rangle| \leqslant 2\varepsilon$.

Thus we can construct two extractions χ, ψ such that $\langle\{F_{\chi(n)}, G_{\psi(n)}\} - \{F, G\}, \phi\rangle$ converges to 0. Since we have $\langle\{F_{\chi(n)}, G_{\psi(n)}\} - H, \phi\rangle \to 0$, it implies $\langle\{F, G\}, \phi\rangle = \langle H, \phi\rangle$, and this equality holds for any ϕ. $\square$

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